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OPTIMAL STRUCTURAL DESIGN OF AN ANNULAR PLATE COMPRESSED BY NON-CONSERVATIVE FORCES

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Abstract—The paper investigates the problem of determining the mass distribution of an annular plate subjected to a compressive follower force, which maximizes its critical load of stability, under the constraint of constant volume. Sensitivity analysis formulae are given for a general formulation of the problem and a computational algorithm is worked out which makes use of them. The results of numerical examples are discussed and some characteristic features of the problem are analysed. The results of both parametric and variational optimization are included.

1. INTRODUCTION

The literature on the optimal design of structural elements under non-conservative loading deals mostly with columns subjected to a follower force. Most of it has been discussed in several survey papers, for example by Błachut and Gajewski (1980), Weisshaar and Plaut (1981), as well as in the monograph by Gajewski and Życzkowski (1988). Recently, Tada *et al.* (1989) formulated the shape determination problem, in which the objective was to maximize the critical load for the Beck column under the condition of constant volume and the condition that the distance between the characteristic curves for adjacent modes was kept wider than a certain value. The shape obtained corresponds to equalization of three subsequent flutter forces (the highest critical load so far).

The optimal structural design of plate elements loaded by non-conservative forces is, as a rule, treated within the framework of aeroelastic problems, and is seen as the minimization of the volume of a plate, given a constant critical velocity of the gas flow. Previous solutions of this problem have been confined to simple panels or rectangular plates. They are presented and discussed, for example, by Pierson (1975) and Seyranian (1982a).

No work has appeared so far investigating the problem of the optimization of an annular plate subjected to non-conservative loading. In the works by Frauenthal (1972) as well as by Grinev and Filippov (1977) only the case of conservative compressive forces is analysed. The effect of follower forces applied on the outer edge of a non-uniform annular plate on its stability and vibration has been investigated in the work by Irie *et al.* (1980), but no attempt to optimize has been made by the authors. In addition the pre-critical state has not been taken into account.

The aim of the present paper is to present a new, and from a theoretical point of view, interesting, optimization problem of an annular plate compressed by uniformly distributed non-conservative forces. Both the pre-critical membrane state and small flexural vibration have been taken into account. In general, the kinetic criterion of stability has to be applied. The formulation of the problem is given for a wide range of boundary conditions. The numerical algorithm is based on sensitivity analysis.

2. THE GOVERNING EQUATIONS OF STATE IN POLAR CO-ORDINATES

We consider an isotropic elastic annular plate loaded on the inner boundary $\bar{r} = \bar{a}$ by $\bar{P}_a$ and on the outer boundary $\bar{r} = \bar{b}$ by $\bar{P}_b$ (Fig. 1a). Here, and in the sequel, symbols with a bar stand for physical quantities, as opposed to non-dimensional ones. The in-plane loadings $\bar{P}_a$ and $\bar{P}_b$ are assumed to be uniformly distributed and to be positive when compressive, the plate thickness $\bar{h}(\bar{r})$ is assumed to be circularly symmetric. Then the pre-critical state is also circularly symmetric and is described by the membrane forces $\bar{N}_r$ and $\bar{N}_\theta$ (positive in the case of tension) and by the radial displacement $\bar{u}_r$. N_r and u_r can be determined from the following linear boundary value problem, written in a non-dimensional form:

$$u' = -\frac{v}{x}u + \frac{1-v^2}{x\Phi}N; \quad N' = \frac{\Phi}{x}u + \frac{v}{x}N, \tag{1}$$

where

$$x = \frac{\bar{r}}{\bar{b}}, \quad N = \frac{x\bar{b}^2}{\bar{D}}\bar{N}_r, \quad u = \frac{\bar{u}}{\alpha\bar{b}}, \quad \bar{D}_0 = \frac{E\bar{h}_0^3}{12(1-v^2)}, \quad \Phi = \frac{\bar{h}}{\bar{h}_0}, \quad \alpha = \frac{\bar{D}_0}{\bar{S}_0\bar{b}^2}, \quad \bar{S}_0 = E\bar{h}_0, \tag{2}$$

and the prime denotes differentiation with respect to x, v is Poisson's ratio, E is Young's modulus, $\bar{h}_0$ is a reference thickness the value of which will be specified in Section 4.

The remaining membrane force $N_\theta = (\bar{b}^2/\bar{D}_0)\bar{N}_\theta$ can be evaluated from the equilibrium condition

$$N_\theta = N'. \tag{3}$$

The boundary conditions relevant to eqns (1) will be assumed in the following general form:

$$[N(\beta)+\pi_1 P]+\kappa_1 u(\beta) = 0, \quad [N(1)+\pi_2 P]+\kappa_2 u(1) = 0, \tag{4}$$

where

$$\beta = \frac{\bar{a}}{\bar{b}}, \quad P_a = \pi_1 P = \frac{\beta\bar{b}^2\bar{P}_a}{\bar{D}_0}, \quad P_b = \pi_2 P = \frac{\bar{b}^2\bar{P}_b}{\bar{D}_0}, \tag{5}$$

the constant parameters κ_1 and κ_2 characterize elastic clampings with respect to the radial displacements of the plate edges, while the parameters π_1 and π_2 determine the ratio of the

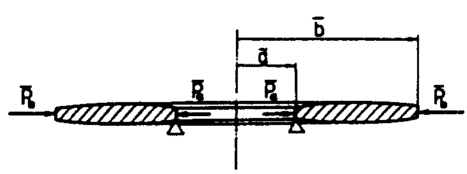

Fig. 1a. Annular plate loaded by uniformly distributed forces of densities $\bar{P}_a$ and $\bar{P}_b$.

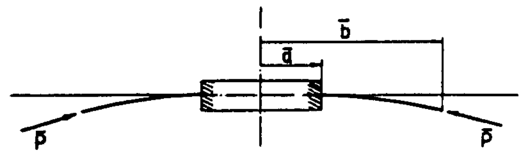

Fig. 1b. The behaviour of the follower load during plate vibration.

loads acting on the outer and inner edges. It has therefore been assumed that the only load parameter is the value of P.

The well-known equation of small vibration superimposed on the pre-critical state of a plate may be transformed to the following set of four ordinary differential equations, as suggested by Grinev and Filippov (1977):

$$
\begin{aligned}
w' &= \phi, \\
\phi' &= \left[\frac{\nu m^2}{x^2} + \frac{N}{2xD}\right] w - \frac{\nu}{x}\phi - \frac{1}{xD} M, \\
M' &= \left[(1-\nu)(3+\nu)\frac{m^2 D}{x^2} + \frac{\Phi u}{2x}\right] w - (1-\nu)(1+\nu+2m^2)\frac{D}{x}\phi + \frac{\nu}{x} M + Q, \\
Q' &= \left[(1-\nu)(2+m^2+\nu m^2)\frac{m^2 D}{x^3} - \frac{N^2}{4xD} + \frac{m^2 \Phi u}{x^2} - \rho\lambda^2 x \Phi\right] w \\
&\quad - \left[(1-\nu)(3+\nu)\frac{m^2 D}{x^2} + \frac{\Phi u}{2x}\right]\phi + \left[\frac{\nu m^2}{x^2} + \frac{N}{2xD}\right] M, \qquad (6)
\end{aligned}
$$

where m is the number of circumferential waves, λ—the non-dimensional frequency of vibration, ρ—the non-dimensional density of the material, and

$$t = \bar{t}/\bar{t}_0, \quad \rho = \bar{\rho}/\bar{\rho}_0, \quad \bar{t}_0 = \left[\frac{\bar{\rho}_0 \bar{h}_0 \bar{b}_0^4}{\bar{D}}\right]^{1/2}, \quad \lambda = \bar{\omega}\bar{t}_0$$

$$\bar{w}(x,t) = \bar{b} w(x)\, e^{i\lambda t} \cos(m\theta)$$

$$\bar{M}_r(x,t) = \frac{\bar{D}_0}{\bar{b}} M_r(x)\, e^{i\lambda t} \cos(m\theta)$$

$$\bar{M}_\theta(x,t) = \frac{\bar{D}_0}{\bar{b}} M_\theta(x)\, e^{i\lambda t} \cos(m\theta)$$

$$\bar{M}_{r\theta}(x,t) = \frac{\bar{D}_0}{\bar{b}} M_{r\theta}(x)\, e^{i\lambda t} \sin(m\theta).$$

M and Q were introduced by Grinev and Filippov (1977), and are defined as follows:

$$M = xM_r + \tfrac{1}{2}Nw, \quad Q = \frac{dM}{dx} + 2mM_{r\theta} - (M_\theta + \tfrac{1}{2}N_\theta w).$$

For a solid plate the stiffness D is related to Φ by the formula

$$D = \Phi^3 \qquad (7)$$

where $D = \bar{D}/\bar{D}_0$.

Equations (6) are analysed under the following general linear and homogeneous boundary conditions:

$$
\begin{aligned}
&\frac{1}{\beta}[M(\beta) - \tfrac{1}{2}N(\beta)w(\beta)] + \kappa_3\phi(\beta) + \alpha_1\pi_1 P w(\beta) + \beta_1\pi_1 P\phi(\beta) = 0 \\
&\frac{1}{\beta}[Q(\beta) + \tfrac{1}{2}N(\beta)\phi(\beta)] + \kappa_4 w(\beta) + \alpha_2\pi_1 P w(\beta) + \beta_2\pi_1 P\phi(\beta) = 0 \\
&[M(1) - \tfrac{1}{2}N(1)w(1)] + \kappa_5\phi(1) + \alpha_3\pi_2 P w(1) + \beta_3\pi_2 P\phi(1) = 0 \\
&[Q(1) + \tfrac{1}{2}N(1)\phi(1)] + \kappa_6 w(1) + \alpha_4\pi_2 P w(1) + \beta_4\pi_2 P\phi(1) = 0 \qquad (8)
\end{aligned}
$$

where the constants $\kappa_3, \ldots, \kappa_6$ characterize the elastic rigidities of the edges with respect to transversal and rotational displacements and the constants $\alpha_1, \ldots, \alpha_4$ and $\beta_1, \ldots, \beta_4$ specify the behaviour of the loads during vibration.

The boundary value problems (1), (4) and (6), (8) determine the so-called characteristic curves, i.e. the relation between the load parameter, P, and the frequency of vibration, λ. For a flutter load two subsequent frequencies coincide for $\lambda \neq 0$, whereas when $\lambda = 0$ a structure loses stability through divergence. In the case of non-conservative loading both possibilities may occur. This depends not only on the behaviour of the load during vibration but also on the modes of support, the distribution of the plate stiffness and so on.

3. SENSITIVITY ANALYSIS

In order to demonstrate the sensitivity analysis method we give sensitivity analysis formulae for an eigenvalue problem with non-linear pre-critical state. The analysis is restricted to systems which can be described by a set of ordinary differential equations of the form

$$Y_i' = G_i(x, Y_i, \Phi, P), \qquad i = 1, \ldots, I \tag{9}$$

$$Z_j' = A_{j\alpha}(x, Y_i, \Phi, P, \lambda) Z_\alpha, \quad j = 1, \ldots, J \tag{10}$$

with boundary conditions

$$B_k^{(0)}[Y_p(0), P] = 0, \quad B_l^{(1)}[Y_r(1), P] = 0 \tag{11}$$

$$\mu_{m\beta}[P, \lambda, Y_p(0)] Z_\beta(0) = 0, \quad \nu_{n\gamma}[P, \lambda, Y_r(1)] Z_\gamma(1) = 0. \tag{12}$$

Repeated Greek indices indicate summation with ranges according to the context.

To derive the equation of sensitivity the standard method of adjoint variables, as described by Haug *et al.* (1986) or Gajewski and Zyczkowski (1988), is followed. According to this method one calculates full variations of (9), (10) and of (11), (12): multiply the first two of the obtained equations by the adjoint variables χ_τ i ψ_η, respectively, integrate by parts and add the results. As a result, the following equation of sensitivity is obtained:

$$\int_\beta^1 g(x)\delta\Phi \, \mathrm{d}x + C_1 \delta P + C_2 \delta\lambda = 0, \tag{13}$$

where

$$g(x) = \chi_\tau \frac{\partial G_\tau}{\partial \Phi} + \psi_\eta \frac{\partial A_{\eta\alpha}}{\partial \Phi} Z_\alpha$$

$$C_1 = \lambda_\kappa^{(0)} \frac{\partial B_\kappa^{(0)}}{\partial P} + \lambda_\lambda^{(1)} \frac{\partial B_\lambda^{(1)}}{\partial P} + \Lambda_\mu^{(0)} \frac{\partial \mu_{\mu\beta}}{\partial P} Z_\beta(0) + \Lambda_\nu^{(1)} \frac{\partial \nu_{\nu\gamma}}{\partial P} Z_\gamma(1) + \int_\beta^1 \left(\chi_\tau \frac{\partial G_\tau}{\partial P} + \psi_\eta \frac{\partial A_{\eta\alpha}}{\partial P} Z_\alpha \right) \mathrm{d}x$$

$$C_2 = \Lambda_\mu^{(0)} \frac{\partial \mu_{\mu\beta}}{\partial \lambda} Z_\beta(0) + \Lambda_\nu^{(1)} \frac{\partial \nu_{\nu\gamma}}{\partial \lambda} Z_\gamma(1) + \int_\beta^1 \psi_\eta \frac{\partial A_{\eta\alpha}}{\partial \lambda} Z_\alpha \, \mathrm{d}x.$$

The functions χ_τ and ψ_η are the solutions of the following adjoint boundary value problems:

$$\chi_i' = -\chi_\tau \frac{\partial G_\tau}{\partial Y_i} - \psi_\eta \frac{\partial A_{\eta\alpha}}{\partial Y_i} Z_\alpha \tag{14}$$

$$\psi_j' = -A_{\alpha j} \psi_\alpha \tag{15}$$

$$\chi_p(0) = -\lambda_\kappa^{(0)} \frac{\partial B_\kappa^{(0)}}{\partial Y_p(0)} - \Lambda_\mu^{(0)} \frac{\partial \mu_{\mu\beta}}{\partial Y_p(0)} Z_\beta(0), \quad \chi_r(1) = \lambda_\lambda^{(1)} \frac{\partial B_\lambda^{(1)}}{\partial Y_r(1)} + \Lambda_\nu^{(1)} \frac{\partial \nu_{\nu\gamma}}{\partial Y_r(1)} Z_\gamma(1) \tag{16}$$

$$\psi_m(0) = -\Lambda_\mu^{(0)} \mu_{\mu m}, \quad \psi_n(1) = \Lambda_\nu^{(1)} \nu_{\nu n}. \tag{17}$$

By making use of eqns (16) and (17), the Lagrange multipliers $\lambda_\kappa^{(0)}$, $\lambda_\lambda^{(1)}$, $\Lambda_\mu^{(0)}$ and $\Lambda_\nu^{(1)}$ can be expressed in terms of the boundary values of the adjoint variables, and the boundary conditions for the adjoint problem can be obtained (transversality conditions).

4. VARIATIONAL OPTIMIZATION OF AN ANNULAR PLATE

In order to obtain greater symmetry of the subsequent formulae the following variables are introduced:

$$\chi_1 = -N^*, \quad \chi_2 = u^*, \quad \psi_1 = -Q^*, \quad \psi_2 = M^*, \quad \psi_3 = -\phi^*, \quad \psi_4 = w^*. \tag{18}$$

For the particular case of an annular plate the general adjoint equations (14) of the pre-critical state take the following form:

$$(u^*)' = -\frac{\nu}{x}u^* + \frac{(1-\nu^2)}{x\Phi}N^* - \frac{1}{2xD}(M^*w + Mw^* - Nww^*)$$
$$(N^*)' = \frac{\Phi}{x}u^* + \frac{\nu}{x}N^* - \frac{\Phi}{2x}(w\phi^* + w^*\phi) + \frac{m^2\Phi}{x^2}ww^*. \tag{19}$$

The adjoint equations of the critical state, written in terms of w^*, ϕ^*, M^* and Q^*, are identical with (6), which results from the special choice of non-physical quantities M and Q in Section 2. This leads to considerable simplification of the numerical analysis.

The following boundary conditions of the adjoint state can be obtained from the transversality conditions (16) and (17):

$$N^*(\beta) + \kappa_1[u^*(\beta) - \tfrac{1}{2}w(\beta)\phi^*(\beta) - \tfrac{1}{2}w^*(\beta)\phi(\beta)] = 0$$
$$N^*(1) + \kappa_2[u^*(1) - \tfrac{1}{2}w(1)\phi^*(1) - \tfrac{1}{2}w^*(1)\phi(1)] = 0 \tag{20}$$

and

$$\frac{1}{\beta}[M^*(\beta) - \tfrac{1}{2}N(\beta)w^*(\beta)] + \kappa_3\phi^*(\beta) - \beta_2\pi_1 Pw^*(\beta) + \beta_1\pi_1 P\phi^*(\beta) = 0$$
$$\frac{1}{\beta}[Q^*(\beta) + \tfrac{1}{2}N(\beta)\phi^*(\beta)] + \kappa_4 w^*(\beta) + \alpha_2\pi_1 Pw^*(\beta) - \alpha_1\pi_1 P\phi^*(\beta) = 0$$
$$[M^*(1) - \tfrac{1}{2}N(1)w^*(1)] + \kappa_5\phi^*(1) - \beta_4\pi_2 Pw^*(1) + \beta_3\pi_2 P\phi^*(1) = 0$$
$$[Q^*(1) + \tfrac{1}{2}N(1)\phi^*(1)] + \kappa_6 w^*(1) + \alpha_4\pi_2 Pw^*(1) - \alpha_3\pi_2 P\phi^*(1) = 0 \tag{21}$$

which, in general, are different from (4) and (8). It may be observed, however, that the boundary conditions of the adjoint critical state (21) and the original boundary conditions (8) become identical when the following conditions are satisfied:

$$\alpha_1 + \beta_2 = 0, \quad \alpha_3 + \beta_4 = 0. \tag{22}$$

In such a case the boundary value problem (6) and (8) is self-adjoint.

In the particular case of an annular plate the general equation of sensitivity (13) can be written as follows:

$$g(x) = \frac{(1-\nu^2)}{x\Phi^2}NN^* + \frac{1}{x}uu^* + \frac{m^2u}{x^2}ww^* - \frac{u}{2x}(w\phi^* + w^*\phi)$$

$$+\frac{dD}{d\Phi}\left[\frac{1}{xD^2}(M-\tfrac{1}{2}Nw)(M^*-\tfrac{1}{2}Nw^*)+(1-\nu)(2+m^2+\nu m^2)\frac{m^2}{x^3}ww^*\right.$$

$$\left.+\frac{(1-\nu)(1+\nu+2m^2)}{x}\phi\phi^*-(1-\nu)(3+\nu)\frac{m^2}{x^2}(w\phi^*+w^*\phi)\right]-\rho\lambda^2xww^*$$

$$C_1 = \pi_2u^*(1) - \pi_1u^*(\beta) + \tfrac{1}{2}\pi_1[w(\beta)\phi^*(\beta) + w^*(\beta)\phi(\beta)] - \tfrac{1}{2}\pi_2[w(1)\phi^*(1) + w^*(1)\phi(1)]$$

$$+\beta\pi_1[\alpha_1\phi^*(\beta)w(\beta) - \beta_2w^*(\beta)\phi(\beta)] + \beta\pi_1[\beta_1\phi^*(\beta)\phi(\beta) - \alpha_2w^*(\beta)w(\beta)]$$

$$+\pi_2[\beta_4w^*(1)\phi(1) - \alpha_3w(1)\phi^*(1)] + \pi_2[\alpha_4w^*(1)w(1) - \beta_3\phi^*(1)\phi(1)]$$

$$C_2 = -2\rho\lambda\int_\beta^1 x\Phi ww^*\,dx. \tag{23}$$

By setting $\delta\Phi = 0$ and $\delta P/\delta\lambda = 0$ in the above equation, one arrives at the following flutter condition:

$$\int_\beta^1 (x\Phi ww^*)\,dx = 0. \tag{24}$$

The optimization problem is formulated here as the maximization of the critical load under constant volume constraint. It entails, according to the gradient projection method presented by Claudon and Sunakawa (1981), iterational improvement of the plate thickness (design function Φ) according to the formula

$$\Phi^{(n+1)} = \Phi^{(n)} + \varepsilon(x)[\mu_1g_1(x) + \mu_2g_2(x) + \cdots + \Lambda x]$$

$$\mu_1 + \mu_2 + \cdots = 1, \tag{25}$$

where $\varepsilon(x)$ is an arbitrary positive function usually assumed to be constant; $\mu_1, \mu_2, \ldots$ are constants to be determined from the conditions of equalization of the critical flutter and/or bifurcation loads, and where Λ is a constant to be calculated from the following constant volume condition:

$$\int_\beta^1 x\Phi\,dx = 1. \tag{26}$$

The explicit formulae from which the above constants can be calculated, for the unimodal and bimodal formulations, are analogous to those given by Seyranian (1982b). The condition (26) is valid when for a given plate volume $\bar{V}$, $\bar{h}_0$ is assumed as $\bar{V}/(2\pi\bar{b}^2)$.

The condition $\mu_1 + \mu_2 + \cdots = 1$ results from the maximin formulation of the problem, namely maximization of the lowest critical load. The gradient functions $g_1(x)$, $g_2(x)$, ... have to be calculated at critical points of flutter or bifurcation.

5. NUMERICAL EXAMPLES

The numerical analysis was performed for a cantilever plate loaded by a follower force uniformly distributed along the outer edge as shown in Fig. 1b. In all calculations β was set equal to 0.2 and Poisson's ratio, ν, was assumed to be 0.3. The reference density $\bar{\rho}_0$, which was introduced in Section 2, has been assumed to be equal to the plate density $\bar{\rho}$ and as a result $\rho = 1$. All differential equations were solved by the Runge–Kutta–Gill integration method of the fourth order and the interval $[\beta, 1]$ was subdivided by 50 nodal points; the accuracy was checked by repeating the calculations with double the number of nodal points.

The boundary and loading conditions of the analysed cantilever plate are a special case of (4) and (8) if the following substitutions are made:

$$\kappa_1 \to \infty,\quad \kappa_3 \to \infty,\quad \kappa_4 \to \infty,\quad \kappa_2 = 0,\quad \kappa_5 = 0,\quad \kappa_6 = 0,$$
$$\pi_1 = 0,\quad \pi_2 = 1,\quad \alpha_3 = 0,\quad \alpha_4 = 0,\quad \beta_3 = 0,\quad \beta_4 = 1. \tag{27}$$

Using the above values of constants it is possible to write all the general formulae of the preceding sections for the special case of the discussed problem.

5.1. *Parametric optimization*

Parametric optimization was carried out for four different design functions: parabolic, linear, harmonic and exponential. We restrict our analysis here to the parabolic function only, as this led to the greatest increase in the critical load. For this case the thickness was assumed in the form:

$$\Phi(x) = C\left[1-\alpha\left(\frac{x-\beta}{1-\beta}\right)^2\right] \tag{28}$$

where

$$\alpha = 1-\frac{\bar{h}_b}{\bar{h}_a},\quad C = \frac{\bar{h}_a}{\bar{h}_0}.$$

$\bar{h}_a$—plate thickness at $\bar{r} = \bar{a}$, $\bar{h}_b$—plate thickness at $\bar{r} = \bar{b}$.

The constant C has been calculated from the constant volume condition (26) and its value is given in Fig. 2a.

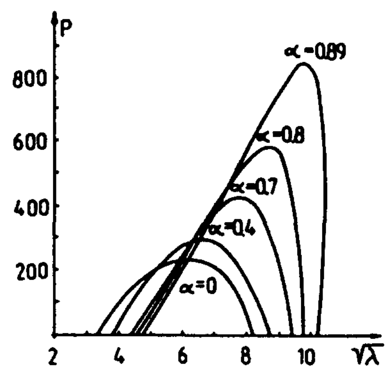

Fig. 2a. Characteristic curves for the first two frequencies for $m = 0$. $\Phi(x) = C\{1-\alpha[(x-\beta)/(1-\beta)]^2\}$, $C = 12/(1-\beta)[6(1+\beta)-\alpha(3+\beta)]$.

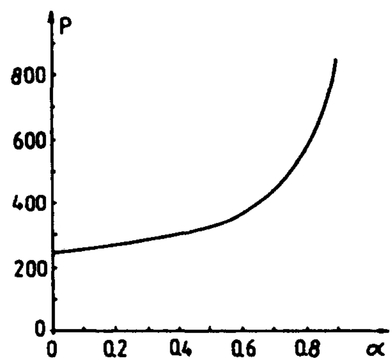

Fig. 2b. Sensitivity of flutter critical load of Fig. 2a with respect to α.

In order to find each point on a characteristic curve the following steps had to be performed:

(a) The eqns (1) were solved with the boundary conditions (4) to find the variables of the pre-critical state;
(b) The eigenvalue problem, (6) and (8), was solved to find a pair (λ, P) and the variables of the critical state. This iterative process was done using the transfer matrix method discussed in some detail by Irie *et al.* (1980).

In Fig. 2a the characteristic curves for $m = 0$, for the first two natural frequencies are shown for different values of the parameter α, and Fig. 2b shows the sensitivity curve of P with respect to α. One feature manifest from these pictures is increasing sensitivity of the flutter load with growing values of α. In fact when optimizing only with respect to $m = 0$, one would arrive at the value $\alpha = 1$ for which $\phi(1) = 0$ and the equations of both pre-critical and critical states would have singularities on the outer edge.

But even though the analysed cantilever plate loses its stability with $m = 0$ when it has constant thickness, it needn't be the case for all thickness distributions. As a consequence, other values of m must also be investigated. The result of such analysis for $\alpha = 0.91$ is shown in Fig. 3. In this picture flutter will first appear not for $m = 0$ but for $m = 4$, and it corresponds to the coalescence of the fifth and sixth natural frequencies for this mode of vibration. By further increasing α it would be possible to increase the flutter load for $m = 0$ but this would be done at the cost of lowering the overall critical load of the plate. Thus the situation shown in Fig. 3 is almost the best result that can be obtained for the assumed design function. It could be improved only slightly by the choice of the parameter α such that the flutter loads for $m = 0$ corresponding to the coalescence of the first and second eigenfrequencies would coincide with that for $m = 4$ and which corresponds to the coalescence of the fifth and sixth natural frequencies. It is noted that when the analysis is performed for different values of m then no singularities arise.

5.2. *Variational optimization*

In variational optimization the following steps were executed, in addition to (a) and (b) of Section 5.1:

(c) The equations of the adjoint critical state, which when written in terms of the starred quantities (18) are the same as (6), were solved with the boundary conditions (21). The transfer matrix in this step was the same as in (b) of Section 5.1. This last fact is a consequence of the following: first, the eigenvalues of an eigenvalue problem and the problem adjoint to it are the same and second, the transfer matrix depends on the coefficients of the differential equations, but not on the boundary conditions;

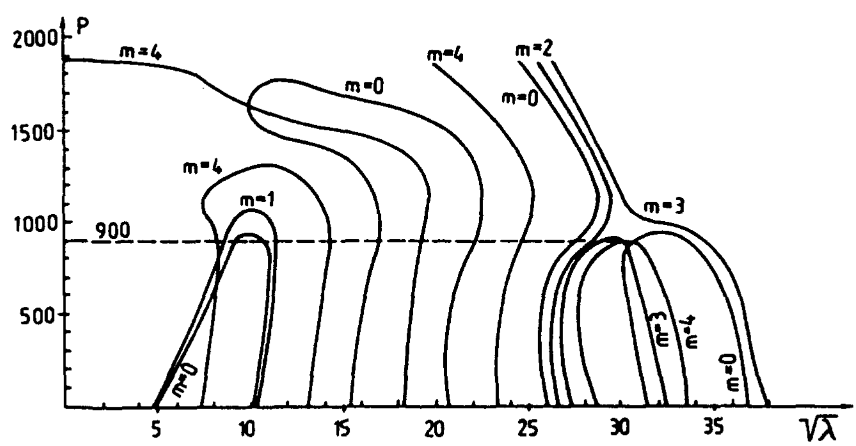

Fig. 3. Selected characteristic curves for the parabolic thickness distribution for $\alpha = 0.91$ and for different values of m and different frequencies.

(d) The equations of the adjoint pre-critical state (19) were solved with the boundary conditions (20), by the shooting method;
(e) The function g and the constant C_1 were calculated from (23);
(f) A new shape was calculated from (25).

The steps (a)–(f) were repeated iteratively. The initial shape was assumed to be prismatic with $\Phi(x) \equiv 2.08333$, which value satisfies (26), and for which the critical flutter load is 243. Using unimodal formulation, in which the gradient was calculated at the flutter point corresponding to the coalescence of the first and second frequencies for $m = 0$, the situation shown in Fig. 4 was reached. For the shape in this figure the critical load $P = 795$ corresponds to $m = 3$ and the plate can lose stability both through divergence and flutter. Again the shape shown in Fig. 4 lies in the region of high sensitivity even though the value of ϕ is nowhere close to 0.

As a result of this sensitivity the pattern of the curves can change very easily. An illustration of this is Fig. 5 which corresponds to a shape almost the same as that in Fig. 4. Even with so small a change in shape there has been a notable increase in the critical load and a change of curve pattern. It can also be seen from Fig. 5 that there are many different modes which lose stability with the value of P close to 812. This fact makes further multimodal optimization difficult. In addition, the justification of further optimization can be questioned if the obtained pattern lies in the sensitive region.

6. CONCLUSIONS

Comparing the results discussed above with the plate of constant thickness an increase in the critical load of 370% has been achieved for parametric optimization and 334% for a variational one. It should be noted, however, that the result obtained by variational optimization could be further improved by multimodal optimization.

As the optimization process depends on the initial shape, it is possible, starting from a non-uniform thickness distribution and using a unimodal optimization procedure, to arrive at the situation when two critical loads coincide but with the load value different from that shown in Fig. 4.

It is pointed out that the solutions of both parametric and variational optimization are very sensitive to small changes in the plate thickness, a difficulty which has been described before in some papers dealing with non-conservative optimization problems. If

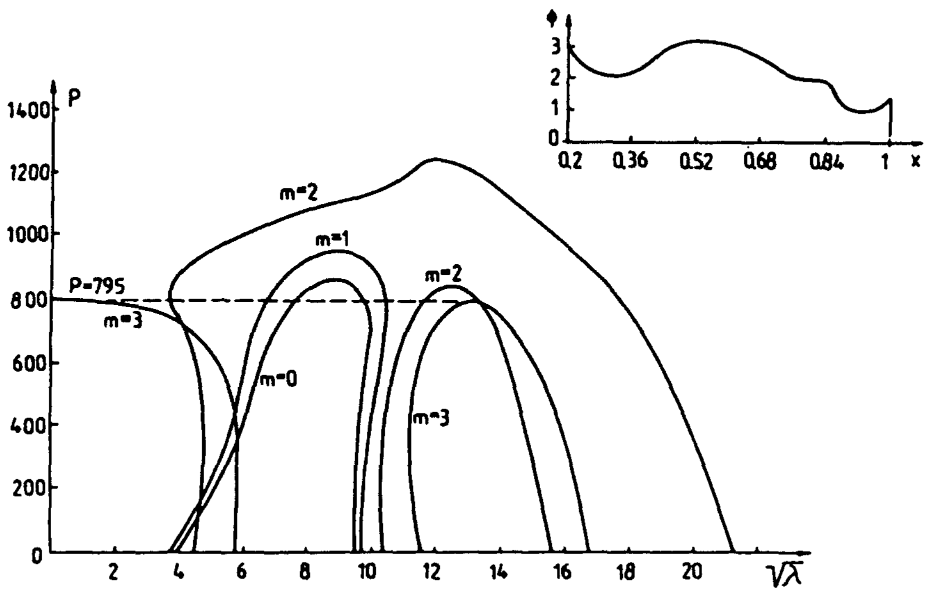

Fig. 4. Shape obtained by unimodal variational optimization and corresponding characteristic curves.

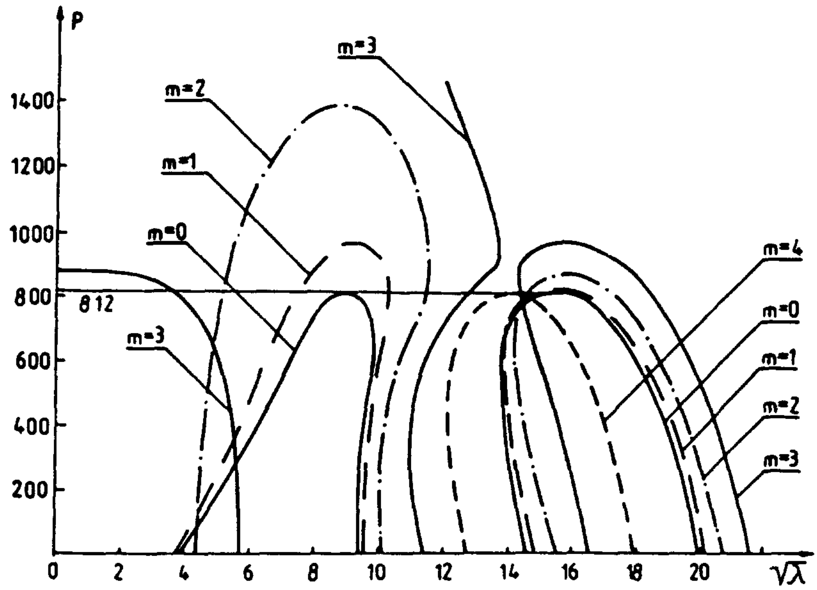

Fig. 5. Characteristic curves for shape close to that shown in Fig. 4.

the optimization process were started not from a uniform thickness distribution but from the best result of parametric optimization, then high sensitivity of the critical load with respect to changes in the plate thickness would appear from the very beginning of the process. As a result the shape changes would be very small even for a notable increase in load values. This behaviour, together with the fact that there exists a different shape obtained by variational optimization process and which is also very sensitive to changes in plate thickness, seems to leave open the question of uniqueness of the solution in the analysed problem.

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